

A generalization of Combinatorial Nullstellensatz

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ABSTRACT. In this note we give an extended version of Combinatorial Nullstellensatz, with weaker assumption on the degree of nonvanishing monomial. We also present an application of our result in a situation where the original theorem does not seem to work.

1. Introduction

The following theorem of Alon, known as Combinatorial Nullstellensatz, has numerous applications in Combinatorics, Graph Theory, and Additive Number Theory (see [1]).

THEOREM 1. (Combinatorial Nullstellensatz [1]) *Let \mathbb{F} be an arbitrary field, and let f be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$. Suppose the coefficient of $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ in f is nonzero and $\deg(f) = \sum_{i=1}^n \alpha_i$. Then for any subsets A_1, \dots, A_n of \mathbb{F} satisfying $|A_i| \geq \alpha_i + 1$, there are $a_1 \in A_1, \dots, a_n \in A_n$ so that $f(a_1, \dots, a_n) \neq 0$.*

In this paper we extend this theorem by weakening the assumption on the degree of nonvanishing monomial. We also provide an explicit formula for coefficients of monomials in the usual expansion of f . Similar results were obtained independently by Schauz [5], however our proofs are simple and more direct. The paper is concluded with an application to a graph labeling problem for which classical approach does not seem to work.

2. Generalized Combinatorial Nullstellensatz

Let \mathbb{F} be an arbitrary field, and let f be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$. We define the *support* of f by $\text{Supp}(f) := \{(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n : \text{the coefficient of } x_1^{\alpha_1} \dots x_n^{\alpha_n} \text{ in } f \text{ is nonzero}\}$. On the set \mathbb{N}^n and hence also on $\text{Supp}(f)$ we have natural partial order: $(\alpha_1, \dots, \alpha_n) \geq (\beta_1, \dots, \beta_n)$ if and only if $\alpha_i \geq \beta_i$ for all i . The proof of the following theorem is a simple extension of an argument found by Michałek [4].

THEOREM 2. (Generalized Combinatorial Nullstellensatz) *Let \mathbb{F} be an arbitrary field, and let f be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$. Suppose $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is a nonvanishing monomial in f and $(\alpha_1, \dots, \alpha_n)$ is maximal in $\text{Supp}(f)$. Then for any subsets A_1, \dots, A_n of \mathbb{F} satisfying $|A_i| \geq \alpha_i + 1$, there are $a_1 \in A_1, \dots, a_n \in A_n$ so that $f(a_1, \dots, a_n) \neq 0$.*

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PROOF. The proof is by induction on $\alpha_1 + \dots + \alpha_n$. If $\alpha_1 + \dots + \alpha_n = 0$, then $f \equiv c \neq 0$ and the assertion is true. If $\alpha_1 + \dots + \alpha_n > 0$, then, without loss of generality, we can assume that $\alpha_1 > 0$. Fix $a \in A_1$ and divide f by $(x_1 - a)$. So, we have

$$f = g \cdot (x_1 - a) + h,$$

where $\deg_{x_1}(h) = 0$. This means that h depends only on the variables x_2, \dots, x_n . If there exists $a_2 \in A_2, \dots, a_n \in A_n$ so that $h(a_2, \dots, a_n) \neq 0$, then we get $f(a, a_2, \dots, a_n) = h(a_2, \dots, a_n) \neq 0$, which proves the assertion. Otherwise $h|_{A_2 \times \dots \times A_n} \equiv 0$. By the division algorithm we have

$$\text{Supp}(g) \subseteq \{(\alpha_1 - r, \alpha_2, \dots, \alpha_n) : (\alpha_1, \alpha_2, \dots, \alpha_n) \in \text{Supp}(f), 1 \leq r \leq \alpha_1\},$$

and $(\alpha_1 - 1, \alpha_2, \dots, \alpha_n) \in \text{Supp}(g)$. Thus the tuple $(\alpha_1 - 1, \alpha_2, \dots, \alpha_n)$ is maximal in $\text{Supp}(g)$. By inductive assumption we know that there exist $a_1 \in A_1 \setminus \{a\}, a_2 \in A_2, \dots, a_n \in A_n$ so that $g(a_1, \dots, a_n) \neq 0$. Hence

$$f(a_1, a_2, \dots, a_n) = (a_1 - a) \cdot g(a_1, \dots, a_n) \neq 0,$$

which proves the assertion of the theorem. \square

3. Coefficient formula

Let \mathbb{F} be an arbitrary field and let A_1, \dots, A_n be any finite subsets of \mathbb{F} . Define the function $N : A_1 \times \dots \times A_n \rightarrow \mathbb{F}$ by

$$N(a_1, \dots, a_n) = \prod_{i=1}^n \prod_{b \in A_i \setminus \{a_i\}} (a_i - b).$$

We may think of the function N as a normalizing factor for the interpolating function on $A_1 \times \dots \times A_n$ defined by

$$\chi_{(a_1, \dots, a_n)}(x_1, \dots, x_n) = N(a_1, \dots, a_n)^{-1} \cdot \prod_{i=1}^n \prod_{b \in A_i \setminus \{a_i\}} (x_i - b).$$

Notice that $\chi_{(a_1, \dots, a_n)}$ is everywhere zero on $A_1 \times \dots \times A_n$, except at the point (a_1, \dots, a_n) for which it takes the value of 1.

We will need the following simple lemma.

LEMMA 1. *Let A be any finite subset of the field \mathbb{F} , with $|A| \geq 2$. Then*

$$\sum_{a \in A} \prod_{b \in A \setminus \{a\}} (b - a)^{-1} = 0.$$

PROOF. Consider the polynomial

$$f(x) = \sum_{a \in A} \prod_{b \in A \setminus \{a\}} \frac{(x - b)}{(a - b)}.$$

Its degree is at most $|A| - 1$, and for all $a \in A$ it takes value of 1. Hence $f \equiv 1$ and the coefficient of $x^{|A|-1}$ equals 0. But it is also the same as the the left hand side of the asserted equality. \square

THEOREM 3. (Coefficient Formula) *Let f be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$ and let $f_{\alpha_1, \dots, \alpha_n}$ denote the coefficient of $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in f . Suppose that $(\alpha_1, \dots, \alpha_n)$ is maximal in $\text{Supp}(f)$. Then for any sets A_1, \dots, A_n in \mathbb{F} such that $|A_i| = \alpha_i + 1$ we have*

$$(*) \quad f_{\alpha_1, \dots, \alpha_n} = \sum_{(a_1, \dots, a_n) \in A_1 \times \dots \times A_n} \frac{f(a_1, \dots, a_n)}{N(a_1, \dots, a_n)}.$$

PROOF. The proof is by induction on the number of elements in the set

$$\text{Cone}(f) = \{\beta \in \mathbb{N}^n : \text{there exists } \alpha \in \text{Supp}(f) \text{ and } \alpha \geq \beta\}.$$

If $|\text{Cone}(f)| = 0$ then $f \equiv 0$ and the theorem is trivial. Otherwise let $(\beta_1, \dots, \beta_n)$ be a maximal element of $\text{Cone}(f)$, so it also belongs to $\text{Supp}(f)$. If $(\beta_1, \dots, \beta_n) = (\alpha_1, \dots, \alpha_n)$, then consider the polynomial

$$f'(x_1, \dots, x_n) = f(x_1, \dots, x_n) - f_{\alpha_1, \dots, \alpha_n} \cdot \prod_{i=1}^n \prod_{b \in A_i \setminus \{a_i\}} (x_i - b)$$

for arbitrary $a_1 \in A_1, \dots, a_n \in A_n$. Notice that

$$\text{Cone}(f') \subset \text{Cone}(f) \setminus \{(\alpha_1, \dots, \alpha_n)\},$$

so from inductive assumption we get the assertion for polynomial f' . Since $(*)$ is \mathbb{F} -linear, to prove it for f it is enough to prove it for the polynomial

$$h = f_{\alpha_1, \dots, \alpha_n}^{-1} \cdot (f - f') = \prod_{i=1}^n \prod_{b \in A_i \setminus \{a_i\}} (x_i - b).$$

But now the equality is obvious: $h(x_1, \dots, x_n) \neq 0$ only for (a_1, \dots, a_n) and therefore $h(a_1, \dots, a_n) = N(a_1, \dots, a_n)$.

If $(\beta_1, \dots, \beta_n) \neq (\alpha_1, \dots, \alpha_n)$ then by the assumptions we have $(\beta_1, \dots, \beta_n) \not\geq (\alpha_1, \dots, \alpha_n)$. So there exists i such that $\beta_i < \alpha_i$, without loss of generality we can assume that $\beta_1 < \alpha_1$. Let $B_1 \subset A_1$ be any subset with β_1 elements. So, we have $|A_1 \setminus B_1| \geq 2$. Consider the polynomial

$$f'(x_1, \dots, x_n) = f(x_1, \dots, x_n) - f_{\beta_1, \dots, \beta_n} \cdot x_1^{\beta_1} \cdots x_n^{\beta_n} \cdot \prod_{b \in B_1} (x_1 - b).$$

As before we have that

$$\text{Cone}(f') \subset \text{Cone}(f) \setminus \{(\beta_1, \dots, \beta_n)\},$$

so, from inductive assumption we get the assertion for polynomial f' . It remains to prove it for the polynomial

$$h = f_{\beta_1, \dots, \beta_n}^{-1} \cdot (f - f') = x_1^{\beta_1} \cdots x_n^{\beta_n} \cdot \prod_{b_1 \in B_1} (x_1 - b_1).$$

Obviously, the right-hand side of equality (*) equals zero. After rewriting the left-hand side we get

$$\begin{aligned}
& \sum_{a \in A_1 \times \dots \times A_n} \left\{ \prod_{i=1}^n \prod_{b \in A_i \setminus \{a_i\}} (b - a) \right\}^{-1} \cdot x_2^{\beta_2} \dots x_n^{\beta_n} \cdot \prod_{b \in B_1} (x_1 - b) = \\
& = \sum_{a_2 \in A_2, \dots, a_n \in A_n} \prod_{i=2}^n \prod_{b \in A_i \setminus \{a_i\}} \left((b - a_i)^{-1} \cdot x_i^{\beta_i} \right) \cdot \\
& \cdot \left(\sum_{a \in A_1} \prod_{b \in A_1 \setminus \{a\}} (b - a)^{-1} \prod_{b \in B_1} (x_1 - b) \right)
\end{aligned}$$

The last factor in this product can be simplified to the form

$$\sum_{a \in A_1 \setminus B_1} \prod_{b \in (A_1 \setminus B_1) \setminus \{a\}} (b - a)^{-1},$$

which is zero by the Lemma 1. The proof is completed. \square

Notice that Theorem 3 implies Theorem 2. Indeed, if $f_{\alpha_1, \dots, \alpha_n} \neq 0$, then f cannot vanish on every point of $A_1 \times \dots \times A_n$. Also if $f_{\alpha_1, \dots, \alpha_n} = 0$, then either f vanishes on the whole set $A_1 \times \dots \times A_n$, or there are at least two points for which f takes a non-zero value.

4. Applications

In this section we give an example of possible application of Theorem 2. In some sense it generalizes the idea of lucky labelings of graphs from [2]. Given a simple graph $G = (V, E)$ and any function $c : V \rightarrow \mathbb{N}$, let $S(u) = \sum_{v \in N(u)} c(v)$ denote the sum of labels over the set $N(u)$ of all neighbors of u in G . The function c is called a *lucky labeling* of G if $S(u) \neq S(w)$ for every pair of adjacent vertices u and w . The main conjecture from [2] states that every k -colorable graph has a lucky labeling with values in the set $\{1, 2, \dots, k\}$. One of the results of [2] in this direction asserts that the set of labels $\{1, 2, 3\}$ is sufficient for every bipartite planar graph G . This result is a special case of the following general theorem.

THEOREM 4. *Let G be a bipartite graph, which has an orientation with outgoing degree bounded by k . Suppose each vertex v is equipped with a non-constant polynomial $f_v \in \mathbb{R}[x]$ of degree at most l and positive leading coefficient. Then there is a labeling $c : V(G) \rightarrow \{1, 2, \dots, kl + 1\}$ such that for any two adjacent vertices u and w ,*

$$c(u) - \sum_{v \in N(u)} f_v(c(v)) \neq c(w) - \sum_{v \in N(w)} f_v(c(v)).$$

PROOF. Assign to each vertex $v \in V(G)$ a variable x_v . Consider the polynomial

$$h = \prod_{uw \in E(G)} \left(\sum_{v \in N(u)} f_v(x_v) + x_w - \sum_{v \in N(w)} f_v(x_v) - x_u \right)$$

We want to show that we can choose values for x_v from the set $\{1, \dots, kl + 1\}$ so that h is non-zero. Let us fix an orientation of G where outgoing degree is bounded by k . For each edge $uw \in E(G)$ oriented $u \rightarrow w$ choose the leading monomial in $f_u(x_u)$ from the factor corresponding to this edge in h . The product of this monomials

is a monomial M of h satisfying $\deg_{x_v}(M) \leq kl$ (since monomials from $f_u(x_u)$ are taken at most k times). We claim that the coefficient of M in h is nonzero. Indeed, each time we take a product of monomials from factors of h resulting in the monomial M , the sign of M is the same (because G is bipartite and leading coefficients of f_v are positive). So the copies of M cannot cancel as we are working in the field \mathbb{R} . Finally, maximality of M in $\text{Supp}(h)$ can be seen easily by collapsing each polynomial $f_v(x_v)$ to x_v . The assertion follows from Theorem 2. \square

Notice that in the above theorem the labels can be taken from arbitrary lists of size at least $kl + 1$.

Let us conclude the paper with the following remark. Suppose that we want to use classical Combinatorial Nullstellensatz to the polynomial $f(x_1, \dots, x_n)$ of degree $\sum_{i=1}^n \alpha_i$ with nonzero coefficient of $x_1^{\alpha_1} \dots x_n^{\alpha_n}$. If $f(x_1, \dots, x_n) = g(h(x_1), x_2, \dots, x_n)$ with $\deg(h) = k$, then for arbitrary sets $A_1, A_2, \dots, A_n \subset \mathbb{F}$, with $h(a) \neq h(b)$ for all distinct $a, b \in A_1$ and of size at least $\alpha_1/k + 1, \alpha_2 + 1, \dots, \alpha_n + 1$, f does not vanish on $A_1 \times \dots \times A_n$. So we gain almost k times smaller first set in comparison with the classical version. It is an immediate consequence of substitution $x'_1 := h(x_1)$ and Theorem 2 applied to $f'(x'_1, x_2, \dots, x_n) = f(x_1, \dots, x_n)$. Analogous corollary is true for more variables being in fact equal to some polynomials.

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